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Investigation of Photon Sphere in Spheroidal Spacetime

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Abstract: We consider the occurrence of black hole from the collapsing perfect fluid configuration in the spheroidal spacetime. In this set up, the particular cases of black hole as an end state of continual collapse with exact solutions for the geometric spheroidal parameter $K = -2$ and $K = -7$ are explored for the formation of photon sphere. It is found that photon sphere does not exist in both the cases, and the divergence of effective potential at the origin rules out formation of a shadow of the black hole.

Keywords: Black hole, photon sphere, shadow of the black hole

1. INTRODUCTION

A lot of attention has been triggered due to recent observations of the shadow of M87 galactic center by the Event Horizon Tele-scope (EHT) collaboration [1], in the subject of general relativis-tic lensing and shadow formation. The observed shadow of this central ultra compact galactic center M87 provides important information of physical parameters spin, mass, etc. and causal structure of spacetime.

In general, shadow is understood to be formed due to the ex-istence of a photon sphere outside the event horizon of a black hole. There are many authors who have compared the shadow of a black hole with the shadow of other compact objects (e.g. naked singularity, gravastar, etc.) [2, 3, 4, 5, 6, 7, 8, 9, 10]. R. Shaikh e. al. [11], have shown that a naked singularity space-time (Joshi-Malafarina-Narayan spacetime) [12] can cast a sim-ilar type of shadow which is expected to be seen in a black hole spacetime. In recent times, Paithankar and Kolarkar stud-ied black hole and acceleration bounds for spherically sym-metric spacetimes [13], and shadow and rings of a de Sitter-Schwarzschild black hole are investigated by Zi-Liang Wang [14]

Vaidya and Tikekar [15, 16, 17, 18] suggested use of com-pact 3-spheroidal geometry to the interior physical space of the compact relativistic stars. This approach admit models of com-pact stars with mass exceeding the limiting value of 3.20 $M_\odot$. The limiting maximum mass 3.20 $M_\odot$ for Neutron stars is ob-tained by Rhodes and Ruffini [19]. The massive compact stars exceeding this limit of 3.20 $M_\odot$ may be unstable and may col-lapse when their equilibrium is lost. These aspects are studied in [20].

So our motivation arise from the question that what if the initial collapsing perfect fluid configuration is spheroidal than the spherical one, whether the geometric spheroidal parame-ters shall contribute/negate the existence of photon sphere, and thereof the shadow of the black hole. We explore the answer to this question in this article.

This article is organised as follows: Section 1 is devoted to the introduction of the topic, its importance and motivation for the work undertaken. In section 2, the work of Sarwe and Tikekar [20] is specified wherein the relativistic equations gov-erning the non-adiabatic shear free collapse of massive super-dense stars in the presence of dissipative forces with associated physical 3-spaces having the 3-spheroidal geometry have been set up. And in the particular models $K = -2$ and $K = -7$ as per the initial data in Table 1, the evolution of collapse ends in the formation of the black hole. The effective potential of pho-tons are studied in section 3 with regard to the formation of the photon sphere. The discussion and conclusions are specified in section 4.

2. BLACK HOLE IN SPHEROIDAL CON-FIGURATION

The interior space-time of a super dense star in accordance with the Vaidya-Tikekar approach is described by the metric [17, 18, 20]

$$ds^2 = -e^{\nu(t,r)}dt^2 + R(t)^2 \left\{ \frac{1-Kr^2}{1-r^2}dr^2 + r^2 d\Omega^2 \right\} \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a 2-sphere. $R(t)$ can be interpreted as a scale factor of an inhomogeneous spherically symmetric 3-space and it is a real valued decreasing function of time with range [0, 1] for a collapsing star. The boundary radius of the configuration at any epoch after the collapse sets in will be $r_s R(t) = r_\Sigma$. The physical 3-spheroidal space is obtained as $t = const.$ section of the space-time which is regular for $r < 1$ and $K < 1$, and immersed in a 4-dimensional Euclidean space.

The interior space-time (1) describing the collapsing cloud configuration becomes static when the continual collapse be-gins with $R(t) = 1$ and ends with $R(t) = 0$, and that the metric potential $e^{\nu(r)}$ is determined by the pressure isotropy condition [18, 20]. The end state phenomenon for choices of K and charac-teristics of photon sphere and shadows thereof can be studied from the static metric

$$ds^2 = -e^{\nu(r)}dt^2 + R^2 \left\{ \frac{1-Kr^2}{1-r^2}dr^2 + r^2 d\Omega^2 \right\} \quad (2)$$

It is shown in [20] that under the perfect fluid conditions, the end sate of gravitational collapse of a superdense cloud is a black hole for the spheroidal geometric parameter $K = -2$ and $K = -7$ involving the data of the relativistic models. Follow-ing Rhodes and Ruffini [19], the surface density of matter of a neutron star is specified as 2×10^{14} $gms\ cm^{-3}$. We consider two specific models from those proposed in Vaidya and Tikekar

(1982) [17] and Tikekar (1990) [18], for the choices $K = -2$ and $K = -7$, of super dense stars of mass about $3.24M_{\odot}$, boundary radius a of the order of 17-18 kms and examined the nature of evolution of the collapse of stars of these models with the den-sity variation parameter λ = surface density/central density. It is generally believed that neutron stars with masses exceed-ing $3.2M_{\odot}$ will be highly unstable and will collapse when their equilibrium is lost [21].

In Table- 1, the data is specified for geometric parameters- K, R, mass and size parameters M, a, density variation and com-pactification parameters λ, M km/a and the exact solution with the metric variable $e^{\nu/2}$ for the two star models chosen [20].

Table 1. Initial data of two star models conceived from [17, 18]

K	λ	a km	R km	M/a	$M/M_{\odot}$	$e^{\nu/2}$
-2	0.494	18.02	34.45	0.2652	3.24	$0.546Z(1-\frac{4}{9}Z^2)$ $+0.999(1-\frac{2}{3}Z^2)^{3/2}$
-7	0.35	17.33	47.44	0.2764	3.2474	$2.83Z(1-\frac{7}{8}Z^2)^{3/2}$ $-0.71(1-\frac{7}{2}Z^2+\frac{49}{24}Z^4)$

where $Z = \sqrt{(1-r^2)}$.

The time t_{BH} of formation of black hole is obtained in [20]

$$t_{BH} = \frac{1}{\alpha_o}\left(\frac{R_{BH}}{2} + \sqrt{R_{BH}} + \ln(1-\sqrt{R_{BH}})\right). \quad (3)$$

The evolution of collapse ends into the formation of black hole with data expressed in Table-2. The time t_{BH} of formation of black hole, the mass parameter m_{BH} of the black hole formed are determined from evolution equations for the collapsing configuration of the specific models for $K = -2$ and $K = -7$. The mass parameter m_{BH} refers to the system of units, rendering $R(t) = 1$ at the time collapse begins, and further satisfying $2m_{BH} = r_s \times R_{BH}$ at the epoch of formation of trapped surfaces. The physical relativistic properties of the final state of the collapse, namely the boundary radius a_{BH} in kms and the mass of the black hole formed $M_{BH}/M_{\odot}$ in the Solar mass unit are also indicated [20].

the sphere so formed due to these unstable circular light-like geodesics is called the photon sphere. For the occurrence of photon sphere of radius (r_{ph}), we should have

$$V_{eff}(r_{ph}) = \frac{1}{b^2}, V'_{eff}(r_{ph}) = 0, V''_{eff}(r_{ph}) < 0.$$

The photon sphere exist at $r = r_{ph}$ if these conditions are sat-isfied, otherwise there is no photon sphere corresponding to the black hole or naked singularity under study. In the formation of photon sphere, the turning point of the geodesic is an im-portant aspect when light from the background source reaches to nearby black hole region. The turning point r_{tp} of light like geodesic can be found when $V_{eff}(r_{tp}) = 1/b^2_{tp}$. Hence, we can write

$$b_{tp} = \frac{r_{tp}R_{tp}}{e^{\nu/2}} \quad (5)$$

where $R_{tp} > R_{BH}$ since the photon sphere lies outside the Schwarzschild radius of the collapsing cloud, and so is the turning point of light like geodesic associated with photon sphere but the value of R_{tp} should be in the immediate neigh-bourhood of R_{BH}.

The light like geodesics emanating from faraway source, can-not reach to the faraway observer with the impact parameter $b < b_{ph}$ where b_{ph} is the impact parameter corresponding to the photon sphere. These geodesics will be trapped inside the photon sphere and this creates a shadow of radius b_{ph} in the observer's sky. In this case, photon sphere's shadow can be seen in observer's sky. On the other hand, when the photon sphere does not exist in a spacetime and the effective potential diverges at the origin, then such a shadow would not form in that spacetime [2].

Table 2. Data obtained in the prescribed models:

K	λ	$r_s = a/$R	$R_{BH}(t)$	m_{BH}	t_{BH}	a_{BH} Kms	$M_{BH}/M_{\odot}$
-2	0.494	0.523	0.2814	0.0736	- 0.1674	5.0708	1.7192
-7	0.35	0.3653	0.3058	0.0559	- 0.1313	5.2995	1.7973

3. NON-EXISTENCE OF PHOTON SPHERE AND BLACK HOLE SHADOW

The null geodesic in the above space-time (2) in the plane $\theta = \pi/2$ can be written as

$$V_{eff} + \frac{e^{\nu(r)}R^2(1-Kr^2)}{L^2(1-r^2)}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} \quad (4)$$

where the effective potential $V_{eff} = \frac{e^{\nu(r)}}{r^2R^2}$ and the impact pa-rameter $b = L/E$ where L and E are conserved angular mo-mentum and conserved energy per unit rest mass.

The investigation of stable and unstable orbits of photons are studied from the very nature of the effective potential of pho-tons. If the effective potential has a maximum value at some radius then we obtain unstable circular orbits of photons, and

Now in the units of the model from Table 2, we have $r_{BH} = r_s \times R_{BH}$. For $K = -2$, $r_{BH} = 0.1472$, so the value of r_{ph} should lie in the range $0.1472 < r_{ph} < 1$. On solving $V'_{eff}(r_{ph}) = 0$, we obtain only imaginary roots, and also $V''_{eff}(r_{ph}) > 0$ for the en-tire possible range $0.1472 < r_{ph} < 1$. Therefore, photon sphere does not exist in this case.

In the case of $K = -7$, $r_{BH} = 0.1118$, so r_{ph} is expected to be in the range $0.1118 < r_{ph} < 1$. Herein $V'_{eff}(r_{ph}) = 0$ gives only one real positive value $r = 0.9748419576$ but this value is very close to initial value of $r < 1$, so we need to discard this choice also as radius of the photon sphere can't be so close to the initial configuration itself. Also $V''_{eff}(r_{ph}) \geq 0$ for the range

$0.1118 < r_{ph} < 1$. Hence, in this case also photon sphere does not exist.

For the models $K = -2$ and $K = -7$, the values of impact parameter b_{tp} obtained using Table 1. and (5) are

$$b_{tp2} = \frac{0.2814\ r_{tp}}{\left[0.546(1-r_{tp}^2)^{1/2}(\frac{5}{9}+\frac{4}{9}r_{tp}^2)+0.999(\frac{1}{3}+\frac{2}{3}r_{tp}^2)^{3/2}\right]} \tag{6}$$

$$b_{tp7} = \frac{0.3058\ r_{tp}}{\left[2.83(1-r_{tp}^2)^{1/2}(\frac{1}{8}+\frac{7}{8}r_{tp}^2)^{3/2}+1.775-2.485r_{tp}^2-1.44959(1-r_{tp}^2)^2\right]} \tag{7}$$

The plots of these values are described in the figure 1. that expresses increasing nature of b_{tp}. Also $b'_{tp} = 0$ gives imaginary values of r_{tp}, so there does not exist a real r_{tp} where b_{tp} is maximum.

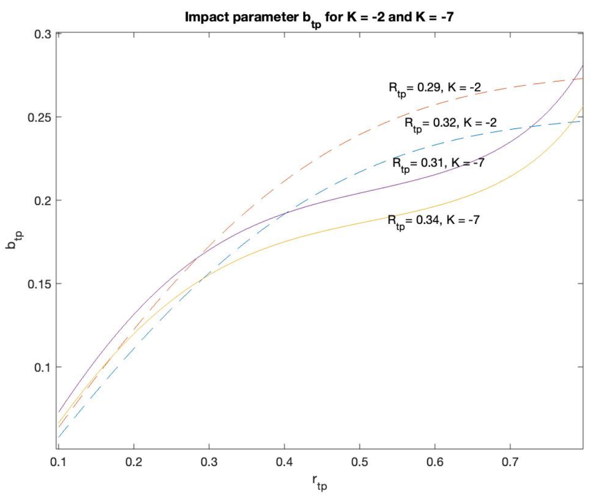

FIGURE 1: The impact parameter b_{tp} is increasing in the range $0.1 < r < 0.8$ indicating that b_{tp} does not have maximum at some r_{tp}.

Next for $K = -2$ and $K = -7$, we compute

$$\lim_{r\to 0} V_{eff} = \lim_{r\to 0} \frac{e^{\nu(r)}}{r^2 R^2} \to \infty$$

Therefore in both the models, there is no shadow of the black hole.

4. DISCUSSION AND CONCLUSIONS

The geometric parameter K of the 3-spheroid is expressed by [17]

$$K = \frac{\left[6\lambda - 1 - \sqrt{1+24\lambda}\right] \mathrm{R}^2}{6\lambda\ a^2} \tag{8}$$

K represents difference in the sphericity of spherical cloud of radius R, giving rise to spheroidal cloud with parameters a and R. The metric coefficient $e^{\nu(r)}$ for $K = -2$ and $K = -7$ ob-tained from solving Einstein's field equations involves traces of λ, a, and R in the end state configuration of black hole. The black hole so formed here is a spindle black hole. In turn, these parameters bring in a change in the effective potential V_{eff} at the turning point of light like geodesic, given by equation (5), thereby attracting/deflecting the rays of light received in the vicinity of the black hole from the background source. Herein, it is found that the change in impact parameter b is insufficient to obtain a radius r_{ph} of photon sphere, so it means the effect of deflection due to spindle geometry is more prominent than the gravitational pull of the black hole, and hence photon sphere does not exist in both the cases $K = -2$ and $K = -7$. Further, the initial geometric spheroidal configuration that col-lapses into its final state of a black hole does not have a shadow as the effective potential at the origin diverges.

This study can be extended to know the effects of gravitational lensing in these two cases.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper

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