Role of Null Condition in Higher Dimensional Gravitational Collapse

R. T. Katre[1*], U. S. Thool[2], R. K. Agrawal[3]
[1]Department of Mathematics, Bajaj College of Science, Wardha, India
[2]Department of Mathematics, Institute of Science, Nagpur, India
[3]Department of Mathematics, Hislop College, Nagpur, India
[*]Corresponding author Email: katrert9@gmail.com

Abstract: In the present work, we investigate the nature of the central singularity arising in the higher dimensional dust collapse. To this end, we solve the radial null geodesic equation in higher dimension using the null condition. We have developed a mathematical approach to generalize the earlier work in four dimensional spacetimes to the five dimensions. It is pointed out that the results on the nature of the outgoing radial geodesics in 4D case can be extended essentially in the same manner in 5D cases also.

Key-words: Gravitational Collapse, Tolmann-Bondi Spacetime, Naked singularity.

1 Introduction

Gravitational collapse is one of the most important and outstanding problems in the classical general relativity. Gravitational collapse has two kinds of possible end states. First is the formation of black holes with outgoing gravitational radiation and matter, and the second one is the formation of naked singularities. It has been proved that, under fairly general hypothesis, solution of the Einstein field equations with physically realistic matter can develop into singularities as a result of gravitational collapse.[1] The main open issue is whether the singularities which arise as the end product of the collapse can actually be observed. According to cosmic censorship hypothesis (CCH) proposed by Penrose,[2] the spacetime singularities produced by gravitational collapse must be covered by the event horizon of the gravity. There are two versions for this hypothesis. The weak version states that a spacetime singularity arising from a generic non-singular initial data is not visible from infinity, whereas the strong version claims that a spacetime singularity developed from non-singular initial data is invisible for any observer - local or faraway. A singularity censored by the strong version is called a naked singularity, while a singularity censored by weal version is called a globally naked singularity. A proof or disproof of the CCH remains an unsolved problem in general relativity. Various models[3-10] on collapse of dust, radiation, perfect fluid etc. studied in recent years, show that either a black holes or a naked singularity form during the gravitational collapse.

The results on the gravitational collapse in higher dimensional spacetimes are important in the view of current possibilities being explored by higher dimensional gravity. As a consequence, the study of gravitational collapse and CCH in higher dimensional spacetimes has now become essential. Many papers on higher dimensional collapse show the occurrence of naked singularities or black holes depending upon the nature of the initial data. [11-15]

In this work, we study the gravitational collapse of Tolman-Bondi spacetime in five dimensions. Patil[16] has studied the occurrence of the naked singularity by root equation method described in Ref. [17]. In this letter, we analyze the nature of the outgoing radial null geodesics by another approach. To this end, we follow the method described in Ref. [18] and solve the radial null geodesic equation using the null condition.

2 Tolmann-Bondi Spacetime in Five Dimension

To facilitate the discussion, we give a brief summary of the five dimensional Tolman-Bondi solution. The inhomogeneous spherically symmetric dust cloud in five dimensional spacetime is given by[16, 19]

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2\left(d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2\, d\theta_3^2\right), \quad (1)$$

where $f(r)$ is an arbitrary function of comoving coordinate r, satisfying $f > -1$, $R(t,r)$ is the physical radius at a time t of the shell labeled by r in the sense that $4\pi R^2$ is the proper area of the shell at time t. Here a prime denotes partial derivative with respect to r.
The energy momentum tensor is given by

$$T_{ij} = \varepsilon\, \delta_t^i \delta_t^j \ , \tag{2}$$

where

$$\varepsilon(t,r) = 3F'/(2R^3\, R')\ . \tag{3}$$

The function $R(t,r)$ is the solution of

$$\dot{R}^2 = F(r)/R^2 + f(r)\ , \tag{4}$$

where the functions $F(r)$ and $f(r)$ are arbitrary and known as mass and energy functions respectively. The over dot denotes partial derivative with respect to t, and here we have set up $8\pi\, G/c^4 = 1$.
We consider the marginally bound case, i.e. $f(r) = 0$. The time $t = t_s(r)$ corresponds to the value $R = 0$, where the area of the shell of a matter at a constant value of coordinate r vanishes. Thus, the ranges of the coordinates for the metric (1) are

$$0 \le r < \infty, \quad -\infty < t \le t_s(r). \tag{5}$$

Since we are concerned with the gravitational collapse, we require that $\dot{R}(t,r) < 0$, hence Eq. (4) yields

$$\dot{R} = -\sqrt{F}/R\ . \tag{6}$$

After integrating the above equation and using the scaling freedom $R(0,r) = r$, we obtain

$$R^2 = r^2 - 2\sqrt{F}\, t\ . \tag{7}$$

As $t_s(r)$ gives the time at which area radius R becomes zero,

$$t_s(r) = r^2/(2\sqrt{F})\ . \tag{8}$$

The Kretschmann scalar $K = R_{abcd}R^{abcd}$ for the metric (1) is given by[16],

$$K = \frac{AF'}{R^6 R'^2} + \frac{B\, F\, F'^7}{R^7 R'} + \frac{C\, F^2}{R^8}\ , \tag{9}$$

where A, B, C are some constants. It can be seen from Eqs. (4) and (9) that the energy and the Kretschmann scalar both diverge at the shell labeled r, indicating the presence of a scalar polynomial curvarure singularity at r.
Singularities in this solution are classified as shell-crossing[20] and shell-focusing[21] singularities. The singularities characterized by $R' = 0$ for $R > 0$ are known as shell-crossing singularities and the one characterized by $R = 0$ are known as shell-focusing singularities. Newman[22] has shown that the shell-crossing singularities are gravitationally weak through which the spacetime may sometimes be extended. Christodoulou[21] has shown that the non-central shell-focusing singularities $(R = 0,\ r > 0)$ are not naked; therefore, we concentrate on the central $(r = 0,\ R = r)$ shell-focusing singularities only.
It follows from the Eq. (4) that the function $F(r)$ becomes fixed once the initial density distribution $\epsilon(0,r) = \rho(r)$ is given, i.e.

$$F(r) = (2/3)\int \rho(r) r^3\, dr\ . \tag{10}$$

We assume that the initial density profile $\rho(r)$ has the series expansion[23]

$$\rho(r) = \rho_0 + \rho_1 r + \frac{\rho_2 r^2}{2!} + \cdots + \frac{\rho_n r^n}{n!} + \cdots, \tag{11}$$

near the center $r = 0$, which on substitution in Eq. (10) yields

$$F = F_0 r^4 + F_1 r^5 + F_2 r^6 + \cdots + F_n r^{n+4} + \cdots, \tag{12}$$

where

$$F_n = \frac{2}{3}\left(\frac{\rho_n}{n!(n+4)}\right). \tag{13}$$

ρ_n being the n^{th} derivative of density at the center and $n = 0, 1, 2, \cdots$. Here we are considering that density function decreases as one moves away from the center. Hence the first non-vanishing derivative in the series expansion (11) should be negative.

The nature of the singularity at the center can be understood by analyzing the behavior of radial null geodesics at the center. The singularity is naked if there exists future directed radial null geodesics in the spacetime with their past end point at the singularity (i.e. at $r = 0$).
Let $u = r^\alpha$, $\alpha > 1$. Then

$$\frac{dR}{du} = \frac{1}{\alpha\, r^{\alpha-1}}\left(\dot{R}\frac{dt}{dr} + R'\right),$$

i.e.
$$\frac{dR}{du} = \frac{R'}{\alpha\, r^{\alpha-1}}\left(1 - \frac{\sqrt{F}}{R}\right) = \frac{R'}{\alpha\, r^{\alpha-1}}\left(1 - \frac{\Lambda}{X}\right) = U(X,u), \tag{14}$$

where $\Lambda = \sqrt{F}/u$, $X = R/u$.

Let us consider the limit X_0 of the tangent along the null geodesic terminating at the singularity at $R = 0$, $u = 0$. Thus

$$X_0 = \lim_{\substack{R\to 0\\ u\to 0}} \frac{R}{u} = \lim_{\substack{R\to 0\\ u\to 0}} \frac{dR}{du} = \lim_{\substack{R\to 0\\ u\to 0}} U(X,u)\,. \tag{15}$$

If a positive real value of X_0 satisfies the above equation the singularity could be naked[16, 17]. If the singularity is naked, some α exists such that at least one finite value of X_0 exists which satisfies the algebraic equation

$$V(X_0) = 0\,, \tag{16}$$

where

$$V(X_0) = U(X,0) - X\,. \tag{17}$$

The above root equation method picks up only the geodesic behaving as $X = R/r^{\alpha} =$ constant. However, there may be the possibility of existence of geodesics which have different behaviors than are assumed. To find such geodesics, we must solve the null geodesic equation[24] using the null condition

$$|\,(dt/dr)/R'| = 1\,. \tag{18}$$

For a particular initial data set to develop either in a naked singularity or a black hole, one has to analyze the behavior of outgoing radial null geodesics, coming from the central singularity $R = 0$, $r = 0$. The outgoing radial null geodesics for the metric (1) are given by

$$dt/dr = R'\,. \tag{19}$$

3 Occurrence of the Naked Singularity

Following Ref. [18], we now discuss the nature of the outgoing radial null geodesics in five-dimensional inhomogeneous dust collapse by considering another approach.

Let us define R(r) as

$$R(r) = a\, r^{\alpha_1}\,, \quad (\alpha_1 > 1). \tag{20}$$

We assume, the expression for mass function F(r) has the form

$$F(r) = F_0 r^4 + F_n r^{4+n} + \text{higher order terms}, \tag{21}$$

where F_n is the first non-vanishing term in the series expansion for F(r) given by Eq. (13).

As we are assuming that the density is decreasing away from the center, the first non-vanishing derivative of the density at the center is negative. Integrating Eq. (6) we obtain

$$R^2/2 = -\sqrt{F}\; t + k\,, \tag{22}$$

where k is a constant of integration. Using the scaling freedom $R(0,r) = r$, above equation yields

$$R^2/\sqrt{F} = r^2/\sqrt{F} - 2t\,. \tag{23}$$

Differentiating the above equation with respect to r and rearranging the terms, we obtain

$$R' = \frac{R\,F'}{4F} + \left(1 - \frac{r\,F'}{4F}\right)\frac{r}{R}\,. \tag{24}$$

Using Eqs. (20) and (21) into Eq. (23) we get,

$$t = \frac{r^2 - a^2 r^{2\alpha_1}}{2\sqrt{F_0 r^4 + F_n r^{n+4}}}\,, \tag{25}$$

which on simplification yields

$$t = \left(\frac{1}{2\sqrt{F_0}}\right) - \left(\frac{F_n}{4F_0^{3/2}}\right) r^n - \left(\frac{a^2 r^{2\alpha_1 - 2}}{2\sqrt{F_0}}\right) + \left(\frac{a^2 F_n}{4F_0^{3/2}}\right) r^{n+2\alpha_1-2}\,. \tag{26}$$

Differentiating the above equation with respect to r and keeping only two lowest order terms we get,

$$\frac{dt}{dr} = -\left(\frac{nF_n}{4F_0^{3/2}}\right) r^{n-1} - \left(\frac{a^2(\alpha_1 - 1)}{\sqrt{F_0}}\right) r^{2(\alpha_1-1)-1}. \tag{27}$$

Inserting the functions R(r) and F(r) into Eq. (24) and simplifying further we obtain

$$R' = ar^{\alpha_1-1} + \left[\left(\frac{nF_n}{4F_0}\right)r^n - \left(\frac{(n+4)F_n^2}{4F_0^2}\right)r^{2n}\right]\left(ar^{\alpha_1-1} - \frac{1}{ar^{\alpha_1-1}}\right),$$

i.e.

$$R' = ar^{\alpha_1-1} - \left(\frac{nF_n}{4aF_0}\right)r^{n+1-\alpha_1}, \quad (28)$$

where we have considered only two lowest ordered terms.
Inserting Eqs. (27) and (28) into Eq. (18), we find that the conditions for the geodesics to be null as $r \to 0$ is that

$$\left|\frac{-\left(\frac{nF_n}{4F_0^{3/2}}\right)r^{n-1} - \left(\frac{a^2(\alpha_1-1)}{\sqrt{F_0}}\right)r^{2(\alpha_1-1)-1}}{ar^{\alpha_1-1} - \left(\frac{nF_n}{4aF_0}\right)r^{n+1-\alpha_1}}\right| = 1. \quad (29)$$

It can be observed that the above null condition will be valid when the numerator and denominator have the equal power of r. This is possible only when $n = 2$ and $\alpha_1 = 2$.
Thus when $n = 2$ and $\alpha_1 = 2$ we have,

$$\left|\frac{\left(\frac{dt}{dr}\right)}{R'}\right| = \left|\frac{\left(\frac{-F_2}{2F_0^{3/2}}\right) - \frac{a^2}{\sqrt{F_0}}}{a - \frac{F_2}{2aF_0}}\right| = 1. \quad (30)$$

Setting $a/\sqrt{(F_0)} = x$ and $F_2/F_0 = \xi$, Eq. (32) becomes

$$2x^3 + 2x^2 + \xi x - \xi = 0. \quad (31)$$

If the above equation has a real and positive root, then it could ensure the existence of the outgoing radial null geodesics emanating from the central singularity, and in this case the singularity could be naked. If this equation has no real and positive root, then it would indicate the absence of outgoing radial null geodesics. In this case, singularity will be covered and the collapse ends into a black hole. Therefore, for the singularity to be naked, Eq. (31) must have a real and positive root. Here we emphasize that, the Eq. (31) is the same equation which has already been obtained in Ref. [16] by another approach.
Numerical evaluation shows that Eq. (31) has a real and positive root if

$$\xi \le -22.18033\ . \quad (32)$$

Thus, for $\xi \le -22.18033$, the central singularity is naked, it is covered for $\xi > -22.18033$. In the analogous four-dimensional case, one gets a quartic equation and the central singularity is naked if and only if $\xi \le -25.9904$.[17] Table 1 shows the real and positive roots of Eq. (31) corresponding to different values of ξ.

ξ	Root x_1	Root x_2
-23	1.4398	1.8583
-24	1.3723	2.0000
-25	1.3292	2.1157
-26	1.2976	2.21157
-27	1.2727	2.3131
-28	1.2523	2.4020
-29	1.2352	2.4863
-30	1.2205	2.5670

Table 1. Roots of the Eq. (31)

When $n < 2$ i.e. $n = 1$ (as n can have only integral values) we can have two cases either $\alpha_1 = 2$ or $\alpha_1 = 1 + n/2 = 3/2$.

Case 1: $\alpha_1 = 2$. In this case, Eq. (30) becomes

$$\left|\frac{\left(\frac{dt}{dr}\right)}{R'}\right| = \left|\frac{\left(\frac{-F_1}{4F_0^{3/2}}\right) - \frac{a^2}{\sqrt{F_0}}r}{ar - \frac{F_1}{4aF_0}}\right| = 1, \quad (33)$$

In the limit of the central singularity (i.e. $r \to 0$), above equation reduces to $|\ a/\sqrt{F_0}\ | = 1$, i.e.

$$a = \sqrt{F_0}\ . \quad (34)$$

Therefore Eq. (20) becomes $R = \sqrt{F_0}\, r^2$. Hence at the central singularity, we get $R^2/F = 1$, i.e.

$$R^2 = F. \tag{35}$$

Thus, in the limit of central singularity, the radial null geodesics obey the law $R^2 = F$. But it can be seen from Eq. (14) that, the equation of apparent horizon (i.e. outer most boundary of the trapped surface) is given by $R^2 = F$. Hence one can assert that, in the limit of the central singularity, null geodesics have a similar behaviour to that of apparent horizon.

Case 2: $\alpha_1 = 3/2$. In this case, Eq. (29) becomes

$$\left|\frac{\left(\frac{dt}{dr}\right)}{R'}\right| = \left|\frac{\frac{-F_1}{4F_0^{3/2}} - \frac{a^2}{2\sqrt{F_0}}}{\left(a - \frac{F_1}{4aF_0}\right)\sqrt{r}}\right| = 1, \tag{36}$$

which implies

$$\frac{F_1}{4F_0^{3/2}} + \frac{a^2}{2\sqrt{F_0}} = 0,$$

i.e.

$$a^2 = -F_1/2F_0. \tag{37}$$

Thus, for this value of a, the null condition (29) is satisfied and hence the singularity is naked. Here we note that as F_1 is the first non-vanishing derivative in the expansion for F, it is negative. It should also be noted that the Eq. (37) is similar to the root equation, which has already been obtained in Ref. [16]

For all other values of α_1, we can observe that the numerator and the denominator in Eq. (29) have unequal powers of r. In addition to this, two terms in R' and two terms in dt/dr have different powers. So, for other values of α_1, null condition (29) cannot be satisfied and hence we cannot have singular null geodesics.

4 Discussion on Entire Family of Singular Geodesics

We follow the method described in Ref. [18] to check whether a family of outgoing null geodesics terminates at the singularity in the past with given root X_0 as tangent. If there is only one outgoing radial null geodesic which terminates at the singularity in the past, then the singularity appears naked only instantaneously to a distant observer. On the other hand, if there is an entire family of radial geodesics then the singularity is to be naked for a finite period of time[25].

Let us assume that, the area radius $R(t, r)$ has the following form.

$$R = X_0 r^{\alpha} + K\, h(r), \tag{38}$$

where K is constant and $h(r)$ is a function of r, which decides the behavior of radial null geodesics.

Consider $\alpha = 1 + n/2$. From Eq. (38) we get

$$\frac{dR}{dr} = \alpha\, X_0 r^{\alpha-1} + K\frac{dh(r)}{dr},$$

which can be written as

$$\frac{dR}{dr} = R' + \dot{R}\frac{dt}{dr}$$

i.e.

$$\frac{dR}{dr} = R'\left(1 - \frac{\sqrt{F}}{R}\right) \tag{39}$$

Using Eq. (20) and (38) with $\alpha = 1 + n/2$, Eq. (28) becomes

$$R' = X_0 r^{n/2} + \left(\frac{Kh(r)}{4}\right) - \left(\frac{nF_n}{4F_0X_0}\right) r^{n/2} + \left(\frac{nF_nK\, h(r)}{4F_0X_0^2\, r}\right). \tag{40}$$

Also, using the binomial theorem, one may write

$$1 - \left(\frac{\sqrt{F}}{R}\right) = 1 - \left(\frac{\sqrt{F_0}}{X_0}\right) r^{1-n/2} + \left(\frac{\sqrt{F_0}}{X_0^2}\right) Kh(r) r^{-n} \tag{41}$$

Inserting the above two equations in Eq.(39), we obtain

$$\alpha\, X_0 r^{\alpha-1} + K\frac{dh(r)}{dr} = \left[X_0\, r^{n/2} + \left(\frac{Kh(r)}{r}\right) - \left(\frac{nF_n}{4F_0X_0}\right) r^{n/2} + \left(\frac{nF_nK\, h(r)}{4F_0X_0^2\, r}\right) K\, h(r)\right] \times \left[1 - \left(\frac{\sqrt{F_0}}{X_0}\right) r^{1-n/2} + \left(\frac{\sqrt{F_0}}{X_0^2}\right) K\, h(r)\, r^{-n}\right] \tag{42}$$

Thus, after satisfying the root of Eq. (31) the differential equation for $h(r)$ becomes

$$K\frac{dh(r)}{dr}=\frac{K\,h(r)}{r}\left[1+\left(\frac{nF_n}{4F_0X_0^2}\right)-\left(\frac{nF_n}{2\sqrt{F_0}\;X_0^3}\right)r^{1-n/2}\right] \tag{43}$$

It can be easily proved from Eq. (17) that the expression in the bracket of the above equation is the same as $\left(1+\frac{n}{2}\right)\frac{dU(X,0)}{dX}\Big|_{X=X_0}$. So, we can write

$$K\frac{dh(r)}{dr}=K\frac{h(r)}{r}\left(1+\frac{n}{2}\right)\frac{dU(X,0)}{dX}\Big|_{X=X_0}$$

i.e.

$$K\frac{dh(r)}{dr}=K\frac{h(r)}{r}\left(1+\frac{n}{2}\right)\left[1+\frac{dV}{dX}\right]_{X=X_0}. \tag{44}$$

We define

$$P_0=1+\frac{dV}{dX}\Big|_{X=X_0}. \tag{45}$$

Let us first consider $n<2,\ i.e.\ n=1\ \ and\ \alpha=3/2$:

From Eq. (37), the root of Eq. (31) is given by

$$X_0^2=-F_1/(2F_0)\,. \tag{46}$$

So from Eq. (43), in the limit of central singularity, i.e. $r\to 0$, we get

$$K\frac{dh(r)}{dr}=K\frac{h(r)}{r}\left[1+\frac{F_1}{4F_0X_0^2}\right]$$

i.e.

$$K\frac{dh(r)}{dr}=K\frac{h(r)}{r}\,, \tag{47}$$

which after integration yields

$$h(r)\propto r^{1/2}. \tag{48}$$

This shows that $h(r)$ goes to zero slower than r. So we have only one radial null geodesic ($K=0$) coming out along this direction. Hence the singularity is visible for an infinitesimal amount of time along this direction.

Eq. (48) can be written in terms of u and P_0 as $h(r)\propto\left(r^{3/2}\right)^{1/3}$, i.e.

$$h(r)\propto u^{P_0}, \tag{49}$$

where $u=r^{\alpha}=r^{3/2}$ and $P_0=1/3$.

Thus, using this method, we have obtained the value of P_0 which is the same as obtained in Ref. [16].

Next consider $n=2$, i.e. $\alpha=2$:

In this case, for the naked singularity, we have two positive roots to Eq. (31). So after cancelling the terms satisfying the root of equation and retaining the lowest power of $h(r)$, we get

$$\frac{dh(r)}{h(r)}=\frac{dr}{r}\left[1+\left(\frac{nF_n}{4F_0X_0^2}\right)-\left(\frac{nF_n}{2\sqrt{F_0}X_0^3}\right)\right], \tag{50}$$

which on integration gives

$$log\,h(r)=\left[1+\left(\frac{nF_n}{4F_0X_0^2}\right)-\left(\frac{nF_n}{2\sqrt{F_0}X_0^3}\right)\right]log\,r \tag{51}$$

i.e.

$$h(r)\ \propto\ r^{\left[1+\left(\frac{nF_n}{4F_0X_0^2}\right)-\left(\frac{nF_n}{2\sqrt{F_0}X_0^3}\right)\right]}. \tag{52}$$

In terms of u and P_0, we write the above equation as

$$h(r)\ \propto\ u^{P_0}, \tag{53}$$

where $u=r^2$ and

$$P_0=\frac{1}{2}\left[1+\left(\frac{F_2}{2F_0X_0^2}\right)-\left(\frac{F_2}{\sqrt{F_0}X_0^3}\right)\right]. \tag{54}$$

Hence for $P_0>1$ there will be a family of radial null geodesics coming out from singularity. With some algebra, it can be shown that

$$P_0=-\frac{1}{4x^3}(\xi+2x^2)\,. \tag{55}$$

Here $x=X_0/\sqrt{F_0}$ and $\xi=F_2/F_0^2$, and we have used the result that x satisfies the cubic Eq. (31).

Since $V(X)=0$ has two positive real roots, it follows that the value of its derivative P_0-1 would be negative along one of the roots and positive along the other. Hence in this situation, family of radial null geodesics will come out from the singularity for which $P_0>1$, while along the other a single will escape. In particular for $\xi=-30$, (satisfying the condition (32)), there are two positive roots to Eq. (31), namely $x_1=1.2205$ and $x_2=2.5670$.

From Eq. (55), we see that $[P_0]_{x=x_1} = 3.7156$ and $[P_0]_{x=x_2} = 0.2486$. Thus along x_1, $P_0 - 1 > 0$ and therefore we have an entire family of radial null geodesics coming out along this direction, whereas along x_2, $P_0 - 1 < 0$ and we have only one radial null geodesics coming out along this direction with $(K = 0)$. The above results are in agreement with the results in Ref. [16].
In the analogous $4D$ case, it has been shown in Ref. [23] that, when the singularity is naked and the parameter $\alpha < 3$, $i.e.$ $n = 1, 2$, there is only one radial null geodesic terminating at the singularity with root X_0 as a tangent. The values of P_0 for $n = 1$ and $n = 2$ were found as 2/5 and 1/7 respectively. For $n = 3$ (in 4D case), P_0 is given by [23]

$$P_0 = -\frac{1}{6x^4}(\xi + 2x^3) ,$$

where $\xi = F_3/F_0^{5/2}$ and $x = X_0/\sqrt{F_0}$, satisfies the quartic equation

$$2x^4 + x^3 + \xi x - \xi = 0.$$

It has also been shown[23] that, if the above quadratic equation has two real and positive roots, the family of geodesics will always terminate along one of the root for which $P_0 > 1$, while along the other, only single geodesic would escape.

5 Discussion on Apparent Horizon

According to Hawking and Ellis,[1] apparent horizon is the boundary of the trapped region. It is known that apparent horizon forms in the region of sufficiently strong gravitational field. An apparent horizon seems to play an important role in deciding the nature of the singularity. It is believed that the formation of the central singularity earlier than apparent horizon is necessary condition for a singularity to be naked. A singularity cannot be naked if it occurs after the formation of an apparent horizon. It has been shown in Ref. [23] that the absence of apparent horizon formation prior to the central singularity does not necessarily imply nakedness. In the present work, we generalized this result to the five dimensional spacetime.
As the density grows without bound, trapped surfaces develop within the collapsing cloud. These can be traced out via the outgoing null geodesics, and the equation of the apparent horizon, $t = t_{ah}(r)$ which makes the outer boundary of the trapped 3-sphere (in 5D case).
For our five-dimensional spacetime it follows from Eq. (14) that trapped surfaces are given by,

$$R(t_{ah}(r), r) = F^{1/2}. \tag{56}$$

Inserting above equation into Eq. (7) we get

$$t_{ah}(r) = \frac{r^2}{2\sqrt{F}} - \frac{\sqrt{F}}{2},$$

i.e.

$$t_{ah}(r) = t_s(r) - \frac{\sqrt{F}}{2}. \tag{57}$$

$F(r)$ is strictly positive for $r > 0$, at $r = 0$, however $t_s(0) = t_{ah}(0)$ and the singularity could be naked.
Let

$$F = F_0 r^4 + F_n r^{4+n} , \tag{58}$$

where F_n is the first non-vanishing term beyond F_0 (note that F_n is negative).
Substituting the expression for F into the Eq. (57) we get,

$$t_{ah}(r) = t_s(0) - \left(\frac{F_n}{4F_0^{3/2}}\right) r^n - \left(\frac{F_0^{1/2}}{2}\right) r^2, \tag{59}$$

where we have kept only the leading order terms. Eq. (59) determines the behaviour of the apparent horizon in the vicinity of the central singularity in five-dimensional spacetime.
Now we consider the different cases according to the first non-vanishing derivative of the density at the centre.
Case (i): Let $\rho_1 \neq 0$ i.e. the first non-vanishing derivative of the density at the centre is ρ_1, then keeping the leading order term in (59) we obtain,

$$t_{ah}(r) = t_s(0) - \frac{F_1}{4F_0^{3/2}} r . \tag{60}$$

Since F_1 is negative, above equation shows $t_s(0) < t_{ah}(r)$, which means central singularity will form earlier than the formation of an apparent horizon. Hence the singularity could be naked in this case.

Case (ii): If the first non-vanishing derivative of the density at the centre is ρ_2 , then we find from Eq. (59) that $t_s(0) < t_{ah}(r)$, if

$$F_2/F_0^2 < -2 \tag{61}$$

i.e. $$t_s(0) < t_{ah}(r)\ ,\ \text{if}\ \ \xi < -2. \tag{62}$$

Thus the central singularity forms earlier than apparent horizon if $\xi < -2$. But according to our previous analysis, singularity is naked if $\xi < -22.18033$. Thus, there is a range of ξ: $-22.18033 < \xi < -2$ in which even though the central singularity forms earlier than apparent horizon, it is not naked.

Case (iii): If the first two derivatives of the density are zero ($\rho_1 = 0,\ \rho_2 = 0$) then $n \geq 3$. So the Eq. (59) becomes (to the leading order)

$$t_{ah}(r) = t_s(0) - \frac{\sqrt{F_0}}{2} r^2\ ,$$

which implies $$t_{ah}(r) < t_s(0),$$

i.e. the apparent horizon forms earlier than the central singularity. Therefore, radial null geodesics cannot escape from the singularity; hence the singularity will appear covered.

We compare the expression $t_{ah}(r) - t_s(0)$ in 4D[23] and 5D cases in Table 2.

n	$t_{ah}(r) - t_s(0)$ *in* 4D	$t_{ah}(r) - t_s(0)$ *in* 5D
$n = 1$	$-\frac{1}{3}\left(\frac{F_1}{F_0^{3/2}}\right) r$	$-\frac{1}{4}\left(\frac{F_1}{F_0^{3/2}}\right) r$
$n = 2$	$-\frac{1}{3}\left(\frac{F_2}{F_0^{3/2}}\right) r^2$	$-\left[\frac{1}{4}\left(\frac{F_2}{F_0^{3/2}}\right) + \frac{\sqrt{F_0}}{2}\right] r^2$
$n = 3$	$-\left[\frac{1}{3}\left(\frac{F_3}{F_0^{3/2}}\right) + \frac{2}{3}F_0\right] r^3$	

Table 2. Comparison of $t_{ah}(r) - t_s(0)$ in 4D and 5D

It can be observed from the table that, the difference $t_{ah}(r) - t_s(0)$ is less in 5D spacetime than the corresponding 4D case. It is quite obvious that, if the apparent horizon forms sufficiently later than the central singularity, then there are maximum chances for radial null geodesics to escape from the singularity. If this difference is reducing then naturally there will be less chances for null geodesics to escape, this may cause decrease in naked singularity spectrum with increase in dimensions.

6 Conclusion

In summary, for the 4D case,[23] there is a range of ξ: $-25.9904 < \xi < -2$, in which even though central singularity forms earlier than apparent horizon, it is not naked. In case of 5D this range reduces to $-22.18033 < \xi < -2$; ($\xi = F_2/F_0^2$). Reduction of this range shows that naked singularity spectrum gets somewhat covered as compare to 4D.

We have discussed the nature of the outgoing radial null geodesics in the vicinity of the central singularity in the five dimensional Tolman-Bondi collapse. To investigate the nature of central singularity, we have solved the radial null geodesic equation using null condition. Assuming that the initial data i.e. F(r) to be finitely differentiable, we have shown that the only first two derivatives of the density function at the centre play the role of deciding the nature of the singularity in the gravitational collapse. We have shown that:

(i) When the first non-vanishing derivative of the density at the centre is ρ_1 (i.e. $n = 1$), there are two possible values of α_1 (in the expression for R(r)) i.e. $\alpha_1 = 2$ or $3/2$. For $\alpha_1 = 3/2$ and $n = 1$ the singularity is naked, whereas for $\alpha_1 = 2$ and $n = 1$ the radial null geodesics have apparent horizon type (i.e. $R^2 = F$) behavior.

(ii) When the first non-vanishing derivative of the density at the centre is ρ_2 (i.e. n = 2), the central singularity is naked if $\xi = F_2/F_0^2 = 2\rho_2/\rho_0^2 \leq -22.18033$. If this inequality is not satisfied, the collapse ends into a black hole. These results are in agreement with the results in Ref. [16].

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