

Two-Fluid Bianchi Type VI Cosmological model with Variable Deceleration Parameter in $f(R, T)$ gravity

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Abstract: In the present paper, we construct Bianchi-type VI cosmological model filled with perfect fluid and dark energy (DE) in $f(R, T)$ theory of gravitation. To obtain the determinate solution of the field equations, the deceleration parameter (q) is assumed to be a function of cosmic time. The proposed work introduces a transitional model that shows the transition from decelerating to accelerating. It is observed that our model reassembles with Λ CDM model. We have also examined the features with cosmic redshift z which are found very close to recent observational data. Additionally, the geometric and physical properties of the cosmological model are discussed.

Keywords: Bianchi type VI model, two fluid, time-dependent deceleration parameter.

1. Introduction:

In recent years, cosmological observations have uncovered that our universe is in the midst of a swift and accelerating expansion, often described as an accelerated expansion. The recent detection of type Ia Supernovae (SNe Ia) [1-8], fluctuations in Cosmic Microwave Background Radiation (CMBR) [9-10], the study of large-scale structures (LSS) [11], data from the Sloan Digital Sky Survey (SDSS) [12-13], Observations from the Wilkinson Microwave Anisotropy Probe (WMAP) [14], data from the Chandra X-ray observatory [15], gravitational lensing [16-17] have collectively offered significant evidence supporting the phenomenon of cosmic acceleration. The phenomenon of cosmic acceleration, which refers to the observed expansion of the universe at an accelerating rate, is one of the most significant and intriguing questions in modern cosmology. To explain this acceleration, several theories and concepts have been proposed. The leading explanation for cosmic acceleration is the existence of a mysterious form of energy called “Dark Energy”. The Cosmic Microwave Background (CMB) anisotropy and LSS of the universe provide compelling evidence that DE is the dominant component of the present universe and is responsible for cosmic acceleration. The WMAP data explains that only 4% of the universe consists of regular matter (Baryonic Matter) and the light it emits, such as stars, planets and gas, approximately 23% is dark matter, a mysterious and invisible substance that doesn’t emit light but exerts gravitational forces and the remaining 73% is composed of DE, an enigmatic force responsible for the universes cosmic accelerated expansion.

DE is an enigmatic factor accountable for the fast expansion of the universe, among other things, cosmological constants [18], quintessence [19-20], equation of state (EoS) parameter [21-24], and interacting dark energy models [25-30] are used by cosmological models to describe the nature of dark energy and the accelerating expansion. The EoS parameter for a perfect fluid in cosmology is defined by a dimensionless quantity denoted as

ω , representing the ratio of its pressure to its density $\omega = \frac{p}{\rho}$ which is closely related to both the thermodynamics

equation of state and the ideal gas. According to current theories, the equation of state parameter $\omega \approx -1$ and at $\omega = -1$, it represents Λ CDM model while for $\omega > -1$ is represents the quintessence model and $\omega < -1$ represents the Phantom model. This parameter does not have to remain constant. It can be expressed as a function of time, the scale factor(a) or redshift (z), indicating its dynamic nature that changes over these varying conditions. An alternative explanation for DE involves modifying the theory of gravity. These modified theories of gravity provide natural explanations for the phenomenon of DE and seek to rationalize the observed cosmic acceleration. The various modified theories are $f(R)$, $f(T)$, $f(G)$, $f(Q)$, and $f(R, T)$ etc.

In this research, we investigated the EoS parameter associated with DE within the spatially homogeneous and anisotropic Bianchi type- VI space-time. This study focused on a configuration involving both a perfect fluid and dark energy, with particular attention given to the impact of a variable deceleration parameter. The $f(R)$ theory suggested that cosmic acceleration can be attained by substituting a general function Ricci scalar $f(R)$, for the Einstein-Hilbert action of general theory relativity. The unification of early-time inflation and late-time

acceleration is demonstrated by feasible $f(R)$ gravity [31]. An extended form of the $f(R)$ gravity theory is presented in the $f(R, T)$ theory, initially proposed by Harko et. al. [32]. In this theory, the gravitational Lagrangian is given by an arbitrary function of the scalar curvature R and the trace T of the energy tensor. The correlation with the trace (T) may arise due to the influence of exotic imperfect fluids or quantum effects within the theoretical framework. The equations of motion for the test particle and the gravitational field equations in the metric formalism are obtained by the author from the energy tensor's covariant divergence of stress. The function $f(R, T)$ has several explicit formulations that correlate with the field equations for the different models that have been presented. Houndjo [33], studied the reconstruction of modified $f(R, T)$ gravity and described the matter-dominated phase and accelerated phase. The FLRW cosmological model has been examined by using the phase space analysis within the framework of $f(R, T)$ gravity by Shabani et.al. [34], Katore et.al. [35] studied in Lyra geometry, anisotropic dark energy cosmological models from early deceleration parameter to late time acceleration. Yadav et. al. [36] investigated a dark energy model within a Bianchi type-III universe characterised by a constant deceleration parameter and in this scenario, EoS parameter is a time-dependent factor. Binary mixture of perfect fluid and dark energy in Bianchi type -I universe was studied by Tade et. al. [37]. Cosmological model Bianchi type VI in the presence of perfect fluid in Branc-Dicke theory studied by Nimkar et. al. [38]. Kaluza-Klein string universe in $f(R, T)$ gravity has been studied by Pawar et.al. [39]

Over the past two decades, extensive research has been conducted on cosmological models with variable deceleration parameters [40-50] and the dynamics of two-fluid systems within the framework of the general theory of relativity, as well as in alternative and modified theories of gravitation. Within the framework of an open FRW space-time containing barotropic and bulk viscous dark fluid the evolution of the DE parameter, has been examined by Amirhaschi, et. al. [51]. The anisotropic behaviour of the accelerating universe in Bianchi V space-time was investigated by Mishra et.al. [52]. Tiwari, et. al. investigated the EoS parameter for DE in a spatially homogeneous and anisotropic Bianchi type-III space-time in the presence of barotropic fluid and DE with variable deceleration parameter [53]. Garg et.al. has presented a transit cosmological model in FRW theory with two fluid scenarios [54]. Two fluid cosmological models investigated by Hatkar, et. al. [55] in scale covariant theories of gravitation. With a non-linear equation of state in the general theory of relativity, Roy et. al. [56] studied a general framework of the emerging universe scenario in interacting and non-interacting fluids.

Given the context discussed above, our aim in this paper is to investigate the Bianchi type-VI cosmological model with variable deceleration parameter, considering non-interacting and interacting scenarios. This paper is organised as follows: Section 1 has been carried out with a foundational overview of the subject matter. Section 2 contains the metric and field equation. The solution of field equations is contained in Section 3. We have discussed various physical and kinematical parameters with non-interacting and interacting cases of the model discussed in Section 4, Section 5 discusses the jerk parameter to explore the validity of our model and Section 6 describes the outcome in brief.

2. The Metric and Field Equations:

We consider the Bianchi Type VI metric as

$$ds^2 = -dt^2 + A^2 dx^2 + e^{-2x} B^2 dy^2 + e^{2x} C^2 dz^2 \quad (1)$$

Where A , B and C are functions of cosmic time t .

For two fluids, the energy-momentum tensor is defined as

$$T_j^i = T_{j(M)}^i + T_{j(D)}^i \quad (2)$$

Where the energy-momentum tensor for perfect fluid is $T_{j(M)}^i$ and $T_{j(D)}^i$ is the energy-momentum tensor of DE These can be written as

$$T_{j(M)}^i = \text{diag}[-\rho_M, p_M, p_M, p_M] = \text{diag}[-1, \omega_M, \omega_M, \omega_M] \rho_M \quad (3)$$

$$T_{j(D)}^i = \text{diag}[-\rho_D, p_D, p_D, p_D] = \text{diag}[-1, \omega_D, \omega_D, \omega_D] \rho_D \quad (4)$$

Where, ρ_M , p_M and $\omega_M = \frac{p_M}{\rho_M}$ are energy density, isotropic pressure and equation of state (EoS) parameter of matter components resp. and ρ_D , p_D and $\omega_D = \frac{p_D}{\rho_D}$ are energy density, isotropic pressure and EoS parameter of DE resp. Here we assume the four-velocity vector $u^i = (1, 0, 0, 0)$ satisfying $u^i u_i = -1$.

The general gravitational field equation in $f(R, T)$ gravity with $f(R, T) = R + 2f(T)$, $f(T) = \mu T$ μ being constant [32], is given by

$$R_{ij} - \frac{1}{2} R g_{ij} = -(8\pi + 2\mu) T_{ij} + (2p\mu + \mu T) g_{ij} \quad (5)$$

In the field equation (5), T is the trace of energy-momentum tensor and p is isotropic pressure in the term $(2p\mu + \mu T)$ and According to [57] and [58], $T = \rho - 3p$.

The above field equation (5) for the metric (1) with the stress-energy momentum tensor (2) gives the following differential equations:

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} + \frac{1}{A^2} = -(8\pi + 2\mu)(p_M + p_D) + \mu(\rho_M - p_M) \quad (6)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} - \frac{1}{A^2} = -(8\pi + 2\mu)(p_M + p_D) + \mu(\rho_M - p_M) \quad (7)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{1}{A^2} = -(8\pi + 2\mu)(p_M + p_D) + \mu(\rho_M - p_M) \quad (8)$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} - \frac{1}{A^2} = (8\pi + 2\mu)(\rho_M + \rho_D) + \mu(\rho_M - p_M) \quad (9)$$

$$\frac{1}{A^2} \left(-\frac{B_4}{B} + \frac{C_4}{C} \right) = 0 \quad (10)$$

Here, suffix 4 denotes the differentiation to time. The law of energy-conservation equation ($T_{;j}^{ij} = 0$) gives,

$$(\rho_M)_{,4} + 3(1 + \omega_M) \rho_M \frac{a_4}{a} + (\rho_D)_{,4} + 3(1 + \omega_D) \rho_D \frac{a_4}{a} = 0 \quad (11)$$

Where a is the average scale factor which is defined with spatial volume as

$$a = \sqrt[3]{ABC} \text{ and } V = a^3 \quad (12)$$

The Hubble Parameter

$$H = \frac{1}{3} \sum_{i=1}^3 H_i = \frac{1}{3} (H_1 + H_2 + H_3) \quad (13)$$

With the help of (12) and (13) we obtain the expansion scalar, shear scalar and mean isotropy respectively as

$$\theta = \frac{A_4}{A} + 2 \frac{B_4}{B} = 3H \quad (14)$$

$$\sigma^2 = \frac{1}{2} \left(\sum_{i=1}^3 H_i^2 - \theta^2 \right) \quad (15)$$

$$A_m = \frac{1}{3} \sum_{i=1}^3 \left(\frac{H_i^2 - H}{H} \right)^2 \quad (16)$$

3. Solutions of the Field Equations:

We have five non-linear differential field equations and seven unknowns namely $A, B, C, \rho_M, p_M, \rho_D$ and p_D . Firstly, on solving equation (10), we get

$$B = k C \quad (17)$$

Where k is the constant of integration but without loss of generality, we take $k = 1$. The ratio of shear σ to Hubble constant H in today's neighbourhood of our Galaxy is place the limit $\frac{\sigma}{H} \leq 0.3$ based on studies of the velocity-red-shift relation for extragalactic sources. Collins et. al. [59] have shown that the normal congruence to the homogeneous expansion for a particularly homogeneous metric satisfies the condition that $\frac{\sigma}{\theta}$ is a constant. To obtain exact solutions we assume the expansion scalar and shear scalar are directly proportional to each other i.e. $\sigma \propto \theta$. This condition leads to [60]

$$A = B^{n_1} \quad (18)$$

Where n_1 is constant.

Now, following Chawla et. al. [41], we have considered the time-dependent deceleration parameter (DP) of the form

$$q = -\frac{a a_{44}}{a_4^2} = b(t) \quad (19)$$

Where a is the average scale factor defined by (10), So the expression of q given in an equation (19) can be written as

$$\frac{a_{44}}{a} + b \frac{a_4^2}{a^2} = 0 \quad (20)$$

By assuming $b = b(a)$ or $b = b(a(t))$, the general solution of (20) is given as

$$\int e^{\int \frac{b}{a} da} da = t + d \quad (21)$$

Where d is a constant of integration. Without loss of generality, we choose $\int \frac{b}{a} da = \log f(a)$, equation (21) can be written in the form

$$\int f(a) da = t + d \quad (22)$$

In equation (22), $f(a)$ the arbitrary function can be taken in such a form that, it will give a physically viable and observationally consistent cosmological model. Thus $f(a)$ is considered as

$$f(a) = \frac{n a^{n-1}}{\gamma \sqrt{1 + a^{2n}}} \quad (23)$$

Where γ is an arbitrary constant & n is a positive constant. Using (23) in (22) with $d = 0$, we get

$$a(t) = \left(\text{Sin } h(\gamma t) \right)^{\frac{1}{n}} \quad (24)$$

Now, by using (10), (16) and (24) we obtained the metric components as follows

$$A = (\text{Sin } h(\gamma t))^{\frac{3n_1}{n(n_1+2)}}$$

$$B = C = (\text{Sin } h(\gamma t))^{\frac{3}{n(n_1+2)}}$$

Hence the given metric can be written in the form

$$ds^2 = -dt^2 + (\text{Sin } h(\gamma t))^{\frac{6n_1}{n(n_1+2)}} dx^2 + (e^{-2x} dy^2 + e^{2x} dz^2) (\text{Sin } h(\gamma t))^{\frac{6}{n(n_1+2)}} \quad (25)$$

4. Calculations and interpretations of other physical and kinematic parameters

For the assumed model the Hubble parameter H , the deceleration parameter q , the expansion scalar θ , the shear scalar σ^2 , volume scale V and the average anisotropy parameter A_m are calculated as follows:

$$V = (\text{Sin } h(\gamma t))^{\frac{3}{n}} \quad (26)$$

$$H = \frac{\gamma}{n} \text{Cot } h(\gamma t) \quad (27)$$

$$\theta = \frac{3\gamma \text{Cot } h(\gamma t)}{n} \quad (28)$$

$$\sigma^2 = \frac{3(n_1 - 1)^2 \gamma^2}{n^2 (n_1 + 2)^2} \text{Cot } h(\gamma t) \quad (29)$$

$$A_m = \frac{3n^2}{(n_1 + 2)^2} - \frac{2(n-1)}{(n_1 + 2)} + 1 = \text{Constant} \quad (30)$$

$$q = n \text{S e } h^2(\gamma t) - 1 \quad (31)$$

Equation (26) – (30) observed that at $t = 0$, the spatial volume (V) vanishes while the other parameters as Hubble parameter (H), expansion scalar (θ) and shear scalar (σ) diverge, shows the model begins with a big bang singularity. Since the anisotropic parameter A_m is constant and non-vanishing, our model maintains its anisotropy as the cosmos expands. Also, the directional scale factors $A(t)$, $B(t)$ and $C(t)$ disappear at the beginning time, this singularity is of the point type (Callum, 1971 [61]).

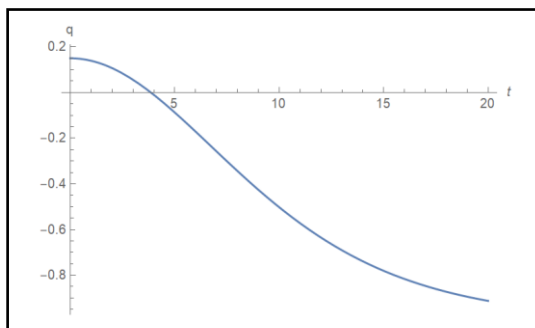


Fig. 1(a) The plot of deceleration parameter versus time t with $\gamma = 0.09804$ and $n = 1.5$

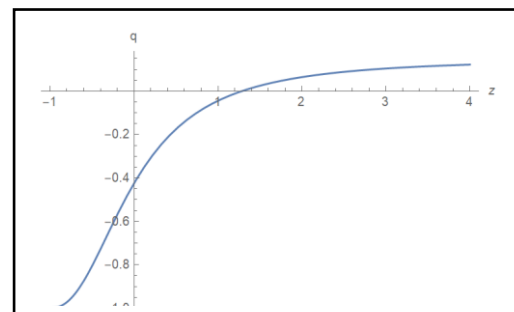


Fig. 1(b) The plot of deceleration parameter versus redshift z with $\gamma = 0.09804$ and $n = 1.15$

Fig.1 and Fig. 2 shows the behaviour of deceleration parameter q with respect to time and redshift. According to SNe Ia's cosmic observations, it too exhibits a phase transition from positive to negative. Moreover, we note that when $t \rightarrow \infty$, $q \rightarrow -1$ and stays negative (accelerating universe) at late times. The cosmos is transitioning from a

decelerating phase ($q > 0$) to an accelerating phase ($q < 0$), as seen by the monotonic decline of the deceleration parameter from a positive value to a negative one. Thus, the latest observations agree with our model [54].

4.1 Non-Interacting Two-Fluid Model

Here, we assume that two fluid models which do not interact with each other. Hence from equation (11), we can write the conservation equation as follows

$$(\rho_M)_{,4} + 3(1 + \omega_M) \rho_M \frac{a_4}{a} = 0 \quad (32)$$

$$\text{And,} \quad (\rho_D)_{,4} + 3(1 + \omega_D) \rho_D \frac{a_4}{a} = 0 \quad (33)$$

On integrating (32), we obtain the value of ρ_M as follows

$$\rho_M = \beta a^{-3(1+\omega_M)} = \beta (\sin h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \quad (34)$$

Where β is an integrating constant.

Since $\omega_M = \frac{p_M}{\rho_M}$, Here we consider ω_M as constant as considered in [21-22]. So, the pressure for matter is obtained as, from (34), we get

$$p_M = \omega_M \beta (\sin h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \quad (35)$$

So, the energy density, isotropic pressure and EoS parameter for DE is defined as follows

$$\rho_D = \frac{1}{8\pi + 2\mu} \left(\frac{9\gamma^2 (2n_1 + 1) \cot h^2 \gamma t}{n^2 (n_1 + 2)^2} - (\sin h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} + (\mu(\omega_M - 3) - 8\pi) \beta (\sin h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \right) \quad (36)$$

$$p_D = \frac{1}{(8\pi + 2\mu)} \left(-\frac{9(n_1^2 + n_1 + 1)}{n^2 (n_1 + 2)^2} \gamma^2 \cot h^2(\gamma t) + \frac{3(n_1 + 1) \gamma^2 \cot h^2(\gamma t)}{n(n_1 + 2)} - \frac{3\gamma^2 (n_1 + 1)}{n(n_1 + 2)} + (\sin h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} + (\mu(1 - 3\omega_M) - 8\pi \omega_M) \beta (\sinh(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \right) \quad (37)$$

$$\omega_D = \frac{\left(-\frac{9(n_1^2 + n_1 + 1)}{n^2 (n_1 + 2)^2} \gamma^2 \cot h^2(\gamma t) + \frac{3(n_1 + 1) \gamma^2 \cot h^2(\gamma t)}{n(n_1 + 2)} - \frac{3\gamma^2 (n_1 + 1)}{n(n_1 + 2)} + (\sin h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} + (\mu(1 - 3\omega_M) - 8\pi \omega_M) \beta (\sin h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \right)}{\left(\frac{9\gamma^2 (2n_1 + 1) \cot h^2 \gamma t}{n^2 (n_1 + 2)^2} - (\sin h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} + (\mu(\omega_M - 3) - 8\pi) \beta (\sin h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \right)} \quad (38)$$

The following figures illustrate how energy density, pressure, and the EoS parameter change with cosmic time. The energy density in Figure 2(a) decreases positively with time and goes to zero as $t \rightarrow \infty$. Figure 2(b) shows that greater redshift z values correspond to a larger universe age. The change in dark fluid pressure p_D as a function of cosmic time and redshift is shown in Figures 3(a) and 3(b), respectively. Pressure p_D is increases and vanishes for the large cosmic time while for redshift z it decreases in negative. The fact that p_D is negative which means the universe is in expanding nature. The transitional behaviour of the EoS parameter with time has been shown in Figure 4(a). The behaviour of ω_D at $\omega_M = 0.5$ shows that it began with the quintessence era and eventually approached the cosmological constant model ($\omega_D = -1$). It is observed that our model reassembles

with Λ CDM model. The EoS parameter changes with redshift parameter z are shown in Figure 4(b), which provides the reliability of the model.

The expression for the matter-energy density parameter and the dark energy density parameter is given by

$$\Omega_M = \frac{n^2 \beta (\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M)}{n}}}{3\gamma^2 \text{Cot } h^2(\gamma t)} \quad (39)$$

$$\Omega_D = \frac{\frac{n^2}{8\pi + 2\mu} \left(\frac{9\gamma^2 (2n_1 + 1) \text{Cot } h^2(\gamma t)}{n^2 (n_1 + 2)^2} - (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1 + 2)}} + (\mu(3 - \omega_M) - 8\pi) \beta (\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \right)}{3\gamma^2 \text{Cot } h^2(\gamma t)} \quad (40)$$

We obtain the total energy density parameter by adding equations (38) and (39),

$$\Omega = \frac{\left(\frac{9\gamma^2 (2n_1 + 1) \text{Cot } h^2 \gamma t}{(n_1 + 2)^2} - n^2 (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1 + 2)}} + n^2 \mu (\omega_M - 1) \beta (\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M)}{n}} \right)}{3\gamma^2 (8\pi + 2\mu) \text{Cot } h^2(\gamma t)} \quad (41)$$

The variation of Ω as a function of cosmic time and redshift is shown in Figures 5(a) and 5(b), respectively. It is observed that Ω is a decreasing function of time.

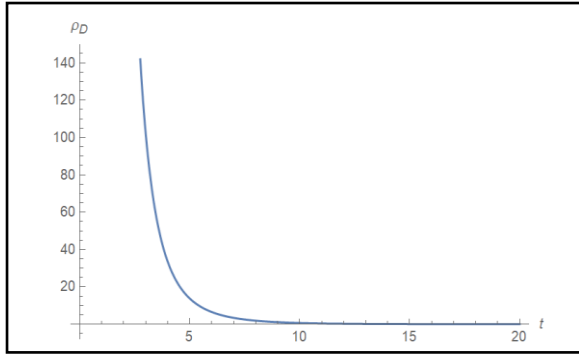


Fig. 2(a) The plot of energy density versus time t

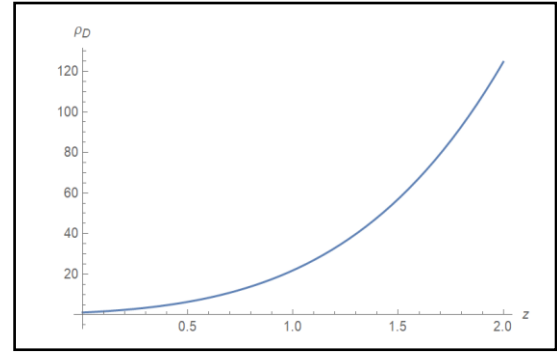


Fig. 2(b) The plot of energy density versus redshift z

with $\gamma = 0.09804$, $n = 1.15$, $n_1 = 5$, $\mu = -13$, $\omega_M = 0.5$, $\beta = 0.001$

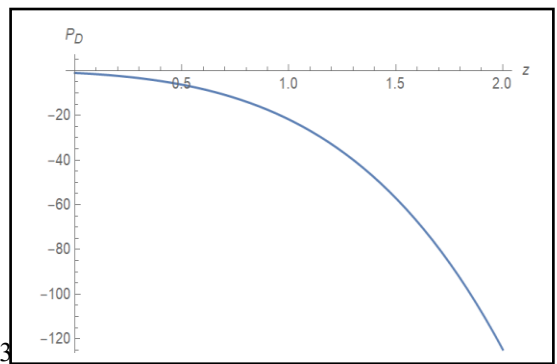
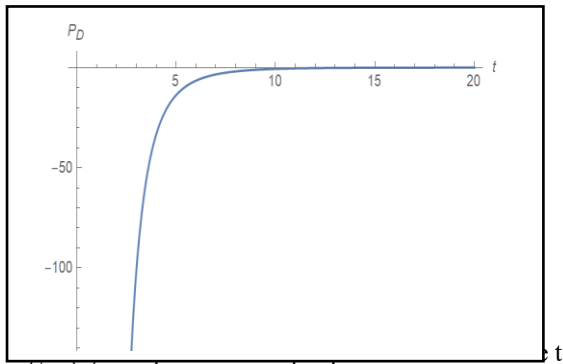


Fig. 3

$\gamma = 0.09804$, $n = 1.15$, $n_1 = 5$, $\mu = -13$, $\omega_M = 0.5$, $\beta = 0.001$

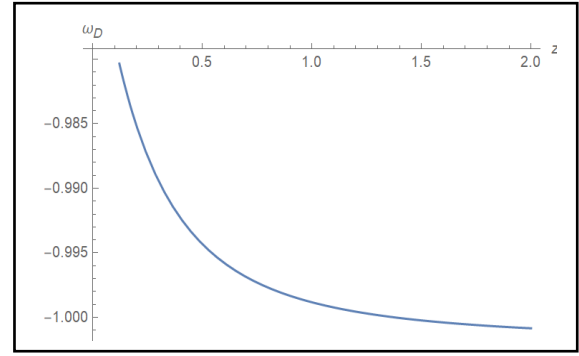
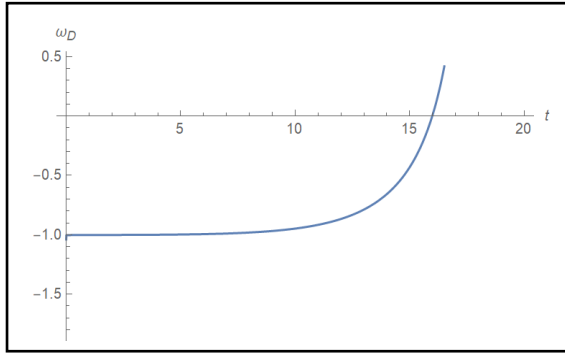


Fig. 4(a) The plot of EoS parameter versus cosmic time t Fig. 4(b) The plot of EoS parameter versus redshift z

$$\gamma = 0.09804, n = 1.15, n_1 = 5, \mu = -13, \omega_M = 0.5, \beta = 0.001$$

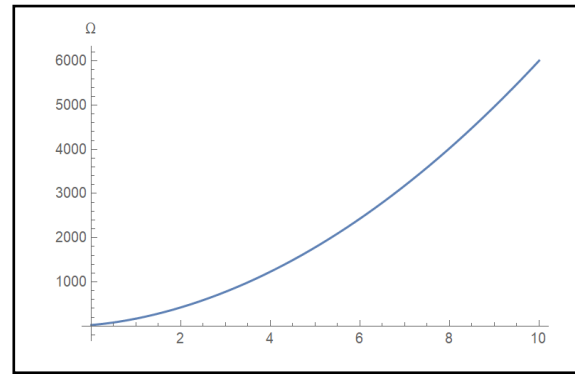
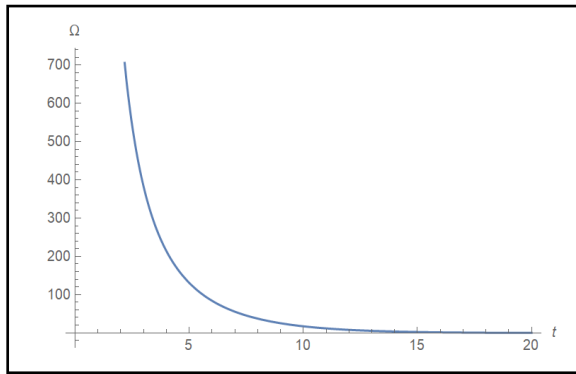


Fig. 5(a) The plot of total energy density versus time t Fig.5(b) The plot of total energy density versus redshift z

$$\gamma = 0.09804, n = 1.15, n_1 = 5, \mu = -13, \omega_M = 0.5, \beta = 0.001, \kappa = 0.02$$

4.2 Interacting Two fluid Model:

When two fluids interact with each other the energy conservation equation becomes

$$(\rho_M)_{,4} + 3(1 + \omega_M) \rho_M \frac{a_4}{a} = - \left((\rho_D)_{,4} + 3(1 + \omega_D) \rho_D \frac{a_4}{a} \right) \quad (42)$$

Consider,

$$(\rho_M)_{,4} + 3(1 + \omega_M) \rho_M \frac{a_4}{a} = Q \quad (43)$$

And thus,

$$\left((\rho_D)_{,4} + 3(1 + \omega_D) \rho_D \frac{a_4}{a} \right) = -Q \quad (44)$$

Here, the interaction between the dark components is expressed by the quantity $Q > 0$. Assuming a positive value of Q , we are considering the scenario where there is a transfer of energy from DE to the matter components and $Q > 0$ guarantees fulfilment of the second law of thermodynamics [62]. Here, it is emphasized that the continuity equations (43) and (44) suggest a proportional relationship between the interaction term (Q) and a quantity possessing units inversely related to time i.e. $Q \propto \frac{1}{t}$. Therefore, a first and natural candidate can be Hubble factor

H multiplied by the energy density. In our model, from Amendola et.al. [63] and Guo et.al. [64], we consider

$$Q = 3H\kappa\rho_M \quad (45)$$

Where κ is coupling constant and it has been found from collaborative analysis of CMB, BAO and SNLS that $-0.08 < \kappa < 0.03$, at 95% CL [64].

Here assuming $\omega_M = \frac{P_M}{\rho_M} = \text{Constant}$.

Using, (45) in (43) and after integrating, we obtain

$$\rho_M = c(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \quad (46)$$

Now using equation (46), we have

$$P_M = c(\omega_M - \kappa)(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \quad (47)$$

Now using (25), (46) and (47) in (9), we have

$$\rho_D = \frac{1}{(8\pi + 2\mu)} \left(\frac{9\gamma^2 (2n_1 + 1)}{n^2 (n_1 + 2)^2} C \cot h^2(\gamma t) - (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} - c(\mu(3 - \omega_M + \kappa) + 8\pi)(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \right) \quad (48)$$

Now, subtracting (8) from (9) and then using equations (25), (46), (47) and (48), we obtain

$$P_D = \frac{1}{(8\pi + 2\mu)} \left(\frac{-9(n_1^2 + n_1 + 1)\gamma^2}{n^2 (n_1 + 2)^2} C \cot h^2(\gamma t) - \frac{3(n_1 + 1)}{n(n_1 + 2)} (1 - C \cot h^2(\gamma t)) + (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} \right. \\ \left. + c(\mu(1 - 3\omega_M + 3\kappa) - 8\pi(\omega_M - \kappa))(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \right) \quad (49)$$

Using (48) and (49), the EoS parameter of dark energy is given by

$$\omega_D = \frac{\left(\frac{-9(n_1^2 + n_1 + 1)\gamma^2}{n^2 (n_1 + 2)^2} C \cot h^2(\gamma t) - \frac{3(n_1 + 1)}{n(n_1 + 2)} (1 - C \cot h^2(\gamma t)) + (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} \right. \\ \left. + c(\mu(1 - 3\omega_M + 3\kappa) - 8\pi(\omega_M - \kappa))(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \right)}{\left(\frac{9\gamma^2 (2n_1 + 1)}{n^2 (n_1 + 2)^2} C \cot h^2(\gamma t) - (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} - c(\mu(3 - \omega_M + \kappa) + 8\pi)(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \right)} \quad (50)$$

The matter-energy density parameter and Dark energy density parameter are defined as

$$\Omega_M = \frac{\rho_M}{3H^2} = \frac{n^2 c (\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}}}{3\gamma^2 C \cot h^2(\gamma t)} \quad (51)$$

$$\Omega_D = \frac{\left(\frac{9\gamma^2 (2n_1 + 1)}{n^2 (n_1 + 2)^2} C \cot h^2(\gamma t) - (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} - c(\mu(3 - \omega_M + \kappa) + 8\pi)(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}} \right)}{3(8\pi + 2\mu) \frac{\gamma^2}{n^2} C \cot h^2(\gamma t)} \quad (52)$$

Adding (51) and (52), we get

$$\Omega = \Omega_M + \Omega_D = \frac{\frac{9\gamma^2 (2n_1 + 1)}{n^2 (n_1 + 2)^2} C \cot h^2(\gamma t) - (\text{Sin } h(\gamma t))^{\frac{-6n_1}{n(n_1+2)}} - c\mu(1 - \omega_M + \kappa)(\text{Sin } h(\gamma t))^{\frac{-3(1+\omega_M-\kappa)}{n}}}{3(8\pi + 2\mu) \frac{\gamma^2}{n^2} C \cot h^2(\gamma t)} \quad (53)$$

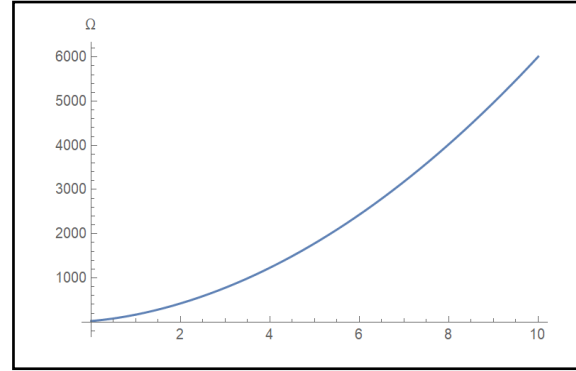
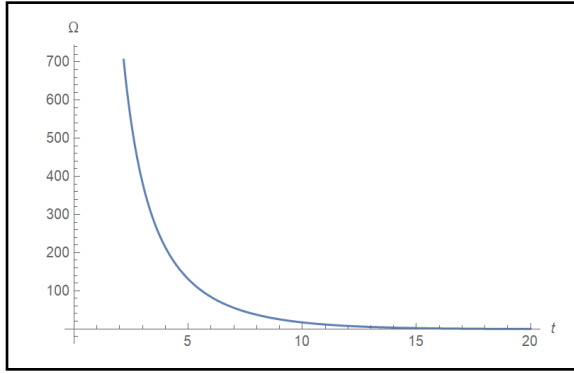


Fig. 6(a) The plot of total energy density versus time t Fig.6(b) The plot of total energy density versus redshift z
 $\gamma = 0.09804, n = 1.15, n_1 = 5, \mu = -13, \omega_M = 0.5, \beta = 0.001, \kappa = 0.02$

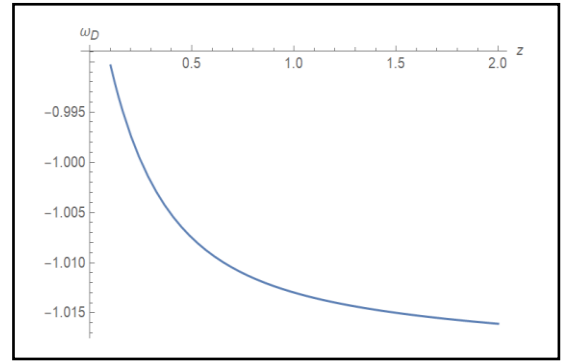
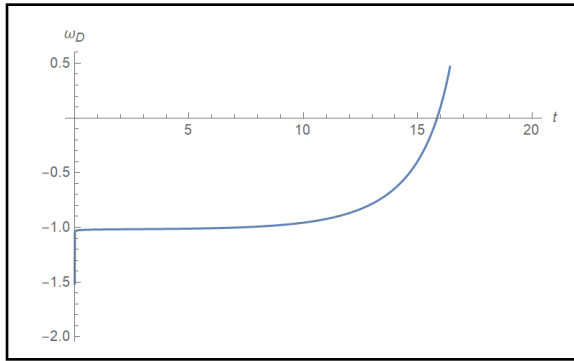


Fig. 7(a) The plot of EoS parameter versus time t

Fig. 7(b) The plot of EoS parameter versus redshift z

$$\gamma = 0.09804, n = 1.15, n_1 = 5, \mu = -13, \omega_M = 0.5, \beta = 0.001, \kappa = 0.02$$

5. The jerk parameter (j):

The third derivative of the scale factor to cosmic time, or Jerk Parameter in cosmology, determines the rate at which the acceleration of the universe's expansion varies over time [65, 66]. It is defined as,

$$j(t) = \frac{\ddot{a}}{aH^3} = \frac{(a^2 H^2)'''}{2H^2} \quad (54)$$

Here, dots represent the derivatives with cosmic time and primes represent the derivatives with scale factors. The jerk parameter appears in the fourth term of a Taylor expansion of the scale factor around a_0

$$\frac{a(t)}{a_0} = 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)^2 + \frac{1}{6}j_0 H_0^3(t-t_0)^3 + O[(t-t_0)^4] \quad (55)$$

Where the subscript represents the present value. Equation (54) can be written in the form

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H} \quad (56)$$

From equation (31) and (56), we write

$$j(t) = n(2n-3) \operatorname{Sec} h^2(\gamma t) + 1 = 1 + \frac{(2n^2 - 3n)}{1 + \left(\frac{1}{z+1}\right)^{2n}} \quad (57)$$

The cosmic jerk parameter in Fig. 8 emphasizes the universe's motion. According to Λ CDM, the universe transits from a decelerated to an accelerated phase in a cosmic jerk j with positive value i.e. $j = 1$ and negative value of q . Initially, at $z = 0$, the jerk parameter attends a positive value less than 1 and approaches 1 in late times.

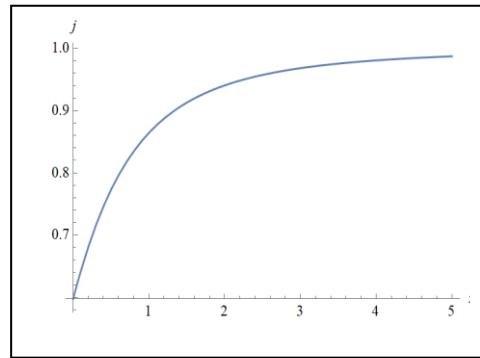


Fig. 8 The plot of jerk parameter versus redshift z

6. Conclusions:

In this paper we have constructed the Bianchi-type- VI cosmological model with two fluid sources i.e. perfect fluid and dark energy source within the framework of $f(R, T)$ gravity. To obtain the deterministic solution we consider the shear-expansion scalar proportionality relation and variable deceleration parameter which represents the transit model for appropriate choices of constants. Also, some physical and kinematical parameters of the constructed model are discussed and from equation (31) it is notice that anisotropic parameter never vanishes so the current model is anisotropic.

Also, it is observed that Current observations and hypotheses are fully consistent with physical parameters like the matter energy density and dark energy density, as well as the pressure in the present model. In both cases i.e. non-interacting and interacting, for $\omega_m < 1$, the variation of ω_d begins from the quintessence region and eventually approaches to cosmological constant region ($\omega_d = -1$) meaning that it never crosses the phantom dividing line. As a result, our model approaches to Λ CDM model and it concluded that, the model exhibits a transition from deceleration to acceleration at low redshift values [67]. In both non-interacting and interacting two-fluid scenarios, the total energy density parameter (Ω) approaches zero after some time, as predicted by current measurements. Moreover, the evolution of energy density, pressure, equation of state parameter and total energy density parameter to redshift parameter z is represented in Fig. 2(b), 3(b), 4(b), 5(b), 6(b) and 7(b) respectively and it is found that the present values are consistent with the observational values. A phase of transition occurs in the universe's expansion from decelerating to accelerating, as predicted by the values of the jerk parameter ($j = 1$) and deceleration parameter ($q = -1$). This transition phase suggests that the model aligns with the Λ CDM model at late times.

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